A Note on Outward and Inward Productions in the Categorical Graph-Grammar Approach and Δ-Grammars

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Abstract

By proving the correspondence between the usual double-pushout approach and Banach's inward version in a purely categorical setting, we can extend the latter to noninjective left-hand sides. In the injective case, Banach's point of view establishes a close relationship between the categorical approach and Kaplan's Δ-grammars allowing a slight generalization of Δ-grammars and making them an operational description of the categorical approach.

1 Introduction

The double-pushout approach to graph grammars uses the pushout construction to generalize the notion of concatenation: A production p is given by a pair of morphisms $p = (B^l \xleftarrow{p^l} K \xrightarrow{p^r} B^r)$ with common domain and derivability is defined by Fig. 1(a). This approach does not refer to special properties of graphs and can easily be extended to other categories that have pushouts. In this paper, we consider an idea that Banach presented recently. In his productions, the arrows go inwards. (Therefore, we use the term outward productions referring to the original form.) The aim of this paper is to generalize Banach's main result both by proving it in a purely categorical setting, thus making it applicable to the case of high-level replacement systems, and by showing that under certain assumptions it also holds for ambiguous derivation steps, i.e., for noninjective p^l. Finally, we establish a correspondence between Banach's inward productions and Kaplan's Δ-productions making the latter an operational interpretation of the double-pushout approach. This result allows intuitively depicting grammars for hierarchically labelled graphs.

As long as graphs are involved, we use the well-known notation: A graph G is a quadruple $G = (E, V, s, t)$ with E and V denoting the set of edges and nodes respectively and $s, t : E \to V$ being mappings. Considering more than one graph, we distinguish their constituents E, V, s and t by indices or

$$\begin{tikzcd}
B^l \arrow[d, "g^l"'] \arrow[dr, phantom, "PO"] & K \arrow[l, "p^l"'] \arrow[r, "p^r"] \arrow[d, "g"] \arrow[dr, phantom, "PO"] & B^r \arrow[d, "g^r"] \\
G^l & C \arrow[l, "\bar{p}^l"] \arrow[r, "\bar{p}^r"'] & G^r
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "p^r"] \arrow[d, "p^l"'] \arrow[dr, phantom, "PO"] & B^r \arrow[d, "q^r"] \\
B^l \arrow[r, "q^l"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
B^l \arrow[r, "q^l"] \arrow[d, "g^l"'] \arrow[dr, phantom, "PO"] & Q \arrow[d, "\bar{g}"] \arrow[dr, phantom, "PO"] & B^r \arrow[l, "q^r"'] \arrow[d, "g^r"] \\
G^l \arrow[r, "\bar{q}^l"'] & D & G^r \arrow[l, "\bar{q}^r"]
\end{tikzcd}$$

(a) (b) (c)

Fig. 1: Fundamental diagrams

apostrophes referring to the denotations of the graphs, e.g., E_G is the set of edges of graph G, etc. A graph morphism f is a pair $f = (f_E, f_V)$ of set morphisms with $f_V s_G = s_H f_E \wedge f_V t_G = t_H f_E$. Limits and colimits in the category $\mathcal{G}raph$ can separately be constructed for nodes and edges in the category $\mathcal{S}et$.

2 Outward vs. inward productions

In formal language theory, a derivation steps starts with looking for a redex $g^l : B^l \to G^l$ and then B^l is "replaced" by B^r. Our double-pushout approach follows this view [3]. In term graph rewriting, however, the steps occur in the reverse order: Gluing some new structure into the graph is done first, then edges are redirected, and finally garbage is removed. In a recently presented lecture, R. Banach could show that this order is also possible in the double-pushout approach [1]:

Definition 2.1 With each outward production $p = (B^l \xleftarrow{p^l} K \xrightarrow{p^r} B^r)$, we associate an inward production $q = (B^l \xrightarrow{q^l} Q \xleftarrow{q^r} B^r)$, where Q is the pushout object in Fig. 1(b).

Definition 2.2 Let $q = (B^l \xrightarrow{q^l} Q \xleftarrow{q^r} B^r)$ be an inward production. We call G^r derivable from G^l via q: $G^l \overset{q}{\Longrightarrow} G^r$ if and only if there exists a morphism $\bar{g} : Q \to D$ such that in Fig. 1(c) both squares are pushouts.

The main result of Banach's lecture is that in the category $\mathcal{G}raph$, we can replace the outward double-pushout construction by an inward one and vice versa if the gluing condition holds and p^l is injective. The first part of this result, however, can be obtained without any restrictions in every category that has pushouts:

Theorem 2.3 *If we have a derivation step $G^l \overset{p}{\Longrightarrow} G^r$ via an outward production p in a category that has pushouts, then $G^l \overset{q}{\Longrightarrow} G^r$ with the associated inward production does also hold.*

Proof. We consider an outward production $p = (p^l, p^r)$ and the inward production $q = (q^l, q^r)$ associated with p (Fig. 2). We assume that the squares $g^l \cdot p^l = \bar{p}^l \cdot g$ and $g^r \cdot p^r = \bar{p}^r \cdot g$ are pushout diagrams, i.e., G^r is derivable from G^l by applying the outward production (p^l, p^r) at redex g^l. Then, we

$$
\begin{tikzcd}
& K \arrow[rr, "p^r"] \arrow[dl, "p^l"'] \arrow[dd, "g"] & & B^r \arrow[dl, "q^r"] \arrow[dd, "g^r"] \\
B^l \arrow[rr, "q^l"'] \arrow[dd, "g^l"'] & & Q \arrow[dd, "\bar{g}"] & \\
& C \arrow[rr, "\bar{p}^r"] \arrow[dl, "\bar{p}^l"'] & & G^r \arrow[dl, "\bar{q}^r"] \\
G^l \arrow[rr, "\bar{q}^l"'] & & D &
\end{tikzcd}
$$

Fig. 2: Pushouts connecting inward and outward derivations

define D by constructing the pushout square $\bar{q}^l \cdot \bar{p}^l = \bar{q}^r \cdot \bar{p}^r$ from $\bar{p}^l$ and $\bar{p}^r$. By definition of Q, the square $q^l \cdot p^l = q^r \cdot p^r$ is a pushout, too. Therefore,

$$(\bar{q}^l \cdot g^l) \cdot p^l = \bar{q}^l \cdot \bar{p}^l \cdot g = \bar{q}^r \cdot \bar{p}^r \cdot g = (\bar{q}^r \cdot g^r) \cdot p^r$$

yields a unique $\bar{g}$ with

$$\bar{q}^l \cdot g^l = \bar{g} \cdot q^l \quad \wedge \quad \bar{q}^r \cdot g^r = \bar{g} \cdot q^r.$$

Now, we use the well-known double-pushout lemma: With $g^r \cdot p^r = \bar{p}^r \cdot g$ and $\bar{q}^r \cdot \bar{p}^r = \bar{q}^l \cdot \bar{p}^l$ being pushouts, $(\bar{q}^r \cdot g^r) \cdot p^r = \bar{q}^l \cdot (\bar{p}^l \cdot g)$ also is a pushout. Furthermore, $(\bar{g} \cdot q^r) \cdot p^r = \bar{q}^l \cdot (g^l \cdot p^l)$ is the same pushout (up to isomorphism) because of $\bar{g} \cdot q^r = \bar{q}^r \cdot g^r \wedge g^l \cdot p^l = \bar{p}^l \cdot g$. Thus, the second half of this pushout, namley $\bar{g} \cdot q^l = \bar{q}^l \cdot g^l$, is a pushout, too. Analogously, we see that $\bar{q}^r \cdot g^r = \bar{g} \cdot q^r$ is a pushout. Putting these pushouts together, we get derivability via q. $\square$

In $\mathcal{G}raph$, this means that if the gluing condition holds in the outward derivation step, then the associated inward production can also be applied, i.e., an analogous gluing condition holds in Banach's approach. If p^l is not injective, the outward production, nevertheless, may be applied if the pushout complement exists, but C and G^r need not be unambiguous. Theorem 2.3 says that at least these graphs G^r can also be derived by the inward production. Conversely, it is easy to prove that every inward derivation step corresponds to an outward one in an HLR1-category [2], if p^l and p^r are in the distinguished class M of morphisms that is typical of an HLR1-category. A more precise discussion, however, shows that a weaker condition is sufficient:

Definition 2.4 Let $\mathcal{C}$ be a class of categories with pushouts such that in each category, there is a distinguished class M of morphisms satisfying the following properties:

(a) If $f : A \to B$ is in M, $g : A \to C$ is any morphism and $q \cdot f = p \cdot g$ is the pushout of (f, g), then p is in M, too.

(b) If p is in M and $p' \cdot p$ has a pushout complement, i.e., there exists a pushout diagram $r \cdot s = p' \cdot p$, then r and s are unambiguous (up to isomorphism).

(c) If in the following diagram both squares are pushout diagrams, p is in M

and $p' \cdot p$ has a pushout complement, then $f' \cdot f$ has a pushout complement, too.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "="] & B \arrow[r, "f'"] \arrow[d, "q"] \arrow[dr, phantom, "="] & B' \arrow[d, "q'"] \\ C \arrow[r, "p"] & D \arrow[r, "p'"] & D' \end{tikzcd}$$

Both $\mathcal{S}et$ and $\mathcal{G}raph$ together with injections as the distinguished class of morphisms satisfy these requirements. In the case of graphs, (b) includes the gluing condition.

Theorem 2.5 *If in a category of class $\mathcal{C}$, p is an outward production with $p^l \in M$, q is the associated inward production and if we have a redex $g^l : B^l \to G^l$, then $G^l \overset{p}{\Longrightarrow} G^r$ holds if and only if $G^l \overset{q}{\Longrightarrow} G^r$.*

Proof. The direction from left to right is a consequence of Theorem 2.3. If on the other hand, $G^l \overset{q}{\Longrightarrow} G^r$ holds, three of the pushouts in the cube of Fig. 2 exist:

$$q^r \cdot p^r = q^l \cdot p^l \qquad \bar{g} \cdot q^l = \bar{q}^l \cdot g^l \qquad \bar{g} \cdot q^r = \bar{q}^r \cdot g^r$$

This means that q^r is in M since p^l is and that $\bar{g} \cdot q^r$ has a pushout complement that is unambiguous. Therefore, the inward derivation step, existence of which we have assumed, yields an unambiguous G^r if the production and the redex are given. From condition (c) in Def. 2.4, we have that $g^l \cdot p^l$ also has a unique pushout complement. Thus we have a unique $\bar{G}^r$ derivable by the outward derivation step. But Theorem 2.3 says that this $\bar{G}^r$ can be derived by an inward step, too. This results in $\bar{G}^r = G^r$. $\square$

This result essentially coincides with Banach's [1]. The advantage of our proof in a purely categorical setting is that we can apply it to more general structures, e.g., high-level replacement systems [2] and hierarchical graphs [5] and that we can analogously prove a symmetrical result allowing ambiguous derivations:

Theorem 2.6 *If in a category of class $\mathcal{C}$, p is an outward production with $p^r \in M$, q is the associated inward production and if we have a redex $g^l : B^l \to G^l$, then $G^l \overset{p}{\Longrightarrow} G^r$ holds if and only if $G^l \overset{q}{\Longrightarrow} G^r$.*

If both p^l and p^r are not in M, however, then the inward production may derive more G^r than the outward one from the same G^l and using the same redex. It is sufficient to give an example in the category $\mathcal{S}et$ (Fig. 3). The elements of the sets are denoted by numbers, which define the mapping as well. (Each element is mapped to the element with the same number; square brackets indicate an element that is the image of more than one element.) Fig. 3(c) gives an example of an inward derivation that is not possible in the outward approach using the same redex g^l. Fig. 4 gives a positive example. (a) depicts the outward, (b) the associated inward production. As before, the nodes are numbered in such a way that the numbers also denote the morphisms: a node is mapped to the node with the same number, e.g., $p^l(1) = 1$.

$$\begin{tikzcd} B^l : \{1,[2,3],4\} & K : \{1,2,3,4\} \arrow[l, "p^l"'] \arrow[r, "p^r"] & B^r : \{[1,2],3,4\} \end{tikzcd}$$

(a) Outward production

$$\begin{tikzcd} B^l : \{1,[2,3],4\} \arrow[r, "q^l"] & Q : \{[1,2,3],4\} & B^r : \{[1,2],3,4\} \arrow[l, "q^r"'] \end{tikzcd}$$

(b) Associated inward production

$$\begin{tikzcd} B^l : \{1,[2,3],4\} \arrow[r, "q^l"] \arrow[d, "g^l"'] & Q : \{[1,2,3],4\} \arrow[d, "\bar{g}"] & B^r : \{[1,2],3,4\} \arrow[l, "q^r"'] \arrow[d, "g^r"] \\ G^l : \{[1,4],[2,3]\} \arrow[r] & D : \{[1,2,3,4]\} & G^r_3 : \{[1,2],[3,4]\} \arrow[l] \end{tikzcd}$$

(c) Inward derivation not possible using outward production

Fig. 3: Counterexample

(In this example, mapping of edges is implicitly defined.) These productions generate the same derivations.

3 Inward productions and Δ-grammars

The operational approaches to graph rewriting describe the process of replacing a subgraph by another one in an immediately implementable way. The Δ-grammars, which were presented by S.M. Kaplan et al. [4], incorporate all relevant items of a production into one tripartite graph:

Definition 3.1 A Δ-production is a tripartite graph

$$\Delta = (E^l \uplus E^c \uplus E^r, V^l \uplus V^c \uplus V^r, s, t)$$

where the following conditions hold:

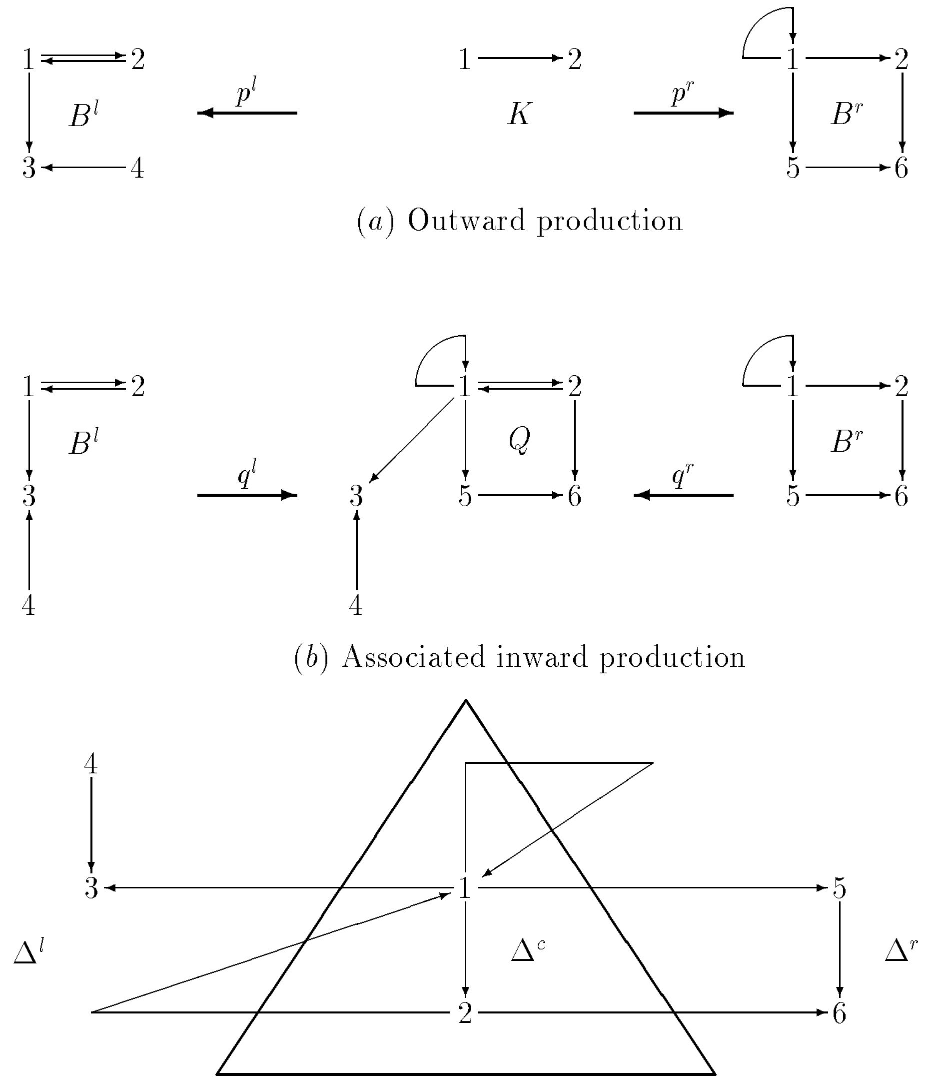
(a) $s, t : E^l \uplus E^r \uplus E^c \to V^l \uplus V^r \uplus V^c$

(b) $e \in E^l \Rightarrow s(e) \notin V^r \wedge t(e) \notin V^r$

(c) $e \in E^r \Rightarrow s(e) \notin V^l \wedge t(e) \notin V^l$

(d) $e \in E^c \Rightarrow s(e) \in V^c \wedge t(e) \in V^c$

Usually, Δ^c is depicted within a triangle, and Δ^l and Δ^r are drawn on the left and on the right, respectively. (Δ^x is shorthand for the nodes V^x and the edges E^x; this is sufficient since the pushout in $\mathcal{G}raph$ is uniquely defined by separately constructing the pushout for nodes and edges.) Δ^l is the fragment of the graph removed during applying the production, Δ^r is the new graph replacing Δ^l. Δ^c denotes a subgraph that is identified, but



(a) Outward production

(b) Associated inward production

(c) Associated Δ-production

Fig. 4: Injective outward production

not changed during rewrite. It is common to both the left-hand side and the right-hand side of the production.[1] (Please note that in general Δ^l and Δ^r are not graphs, but Def. 3.1 ensures that $\Delta^l \uplus \Delta^c$ and $\Delta^c \uplus \Delta^r$ are graphs if we use suitable restrictions of s, t.) Fig. 4(c) completes our example by depicting the Δ-version of the production. Δ^c consists of the nodes within the triangle and the edges not leaving it, whereas an edge a part of which is drawn outside the triangle belongs to Δ^l or to Δ^r, respectively. E.g., the edges from 2 to 1 and from 1 to 3 are part of Δ^l.

Definition 3.2 If a Δ-production is given and we have

$$g^l : (E^l \uplus E^c, V^l \uplus V^c, s_p, t_p) \to G$$

an injective graph morphism where s_p, t_p are the restrictions of s, t to $E^l \uplus E^c$,

[1] Kaplan's productions have two further components, a negative application condition and a textually expressed guard; both are omitted here.

$$
\begin{tikzcd}
\cdot \arrow[r, "p"] \arrow[d, "q"'] & \cdot \arrow[d, "q'"] & \cdot \arrow[l, "\bar{p}"'] \arrow[dl] \\
\cdot \arrow[r, "p'"'] & \cdot &
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
\cdot \arrow[r, "p"] \arrow[d, "q"'] & \cdot \arrow[d, "q'"] \\
\cdot \arrow[r, "p'"] & \cdot \\
\cdot \arrow[u, "g"] \arrow[ur, "g'"'] &
\end{tikzcd}
$$

(b)

Fig. 5: Relationship between pushouts and coproducts

then H is called derivable from G with $\Delta: \; G \overset{\Delta}{\Longrightarrow} H$ if it is constructed as follows:

$$
\begin{aligned}
E_H &:= (E_G \setminus g^l[E^l]) \cup E^r \\
V_H &:= (V_G \setminus g^l[V^l]) \cup V^r \\
s_H(e) &:= \begin{cases} s_G(e) & \text{if } e \in E_G \setminus g^l[E^l] \\ s_p(e) & \text{if } e \in E^r \wedge s_p(e) \in V^r \\ g^l(s_p(e)) & \text{if } e \in E^r \wedge s_p(e) \in V^c \end{cases} \\
t_H(e) & \quad \text{analogously}
\end{aligned}
$$

We can see equivalence between Δ-derivability and double-pushout derivability by using the following lemma that does hold in every category with pushouts and coproducts:

Lemma 3.3 *(a) If $q' \cdot p = p' \cdot q$ and if there exists a $\bar{p}$ such that $(p, \bar{p})$ is a coproduct, then $q' \cdot p = p' \cdot q$ is a pushout diagram if and only if $(p', q' \cdot \bar{p})$ is a coproduct (Fig. 5(a)).*

(b) If p and q' are given such that there exists a g' with (g', q') being a coproduct, then the coproduct (g, q) yields a pushout diagram $q' \cdot p = p' \cdot q$ (Fig. 5(b)).

If a production $\Delta = (\Delta^l, \Delta^c, \Delta^r)$ is given, we can construct an injective outward production $p : \Delta^l \coprod \Delta^c \xleftarrow{p^l} \Delta^c \xrightarrow{p^r} \Delta^c \coprod \Delta^r$ with p^l and p^r being the natural coproduct morphisms. (We use coproduct notation instead of disjoint union to keep discussion more general.) For injective redices, Fig. 6(a) shows that $G \overset{p}{\Longrightarrow} H \Leftrightarrow G \overset{\Delta}{\Longrightarrow} H$ is an immediate consequence of Lemma 3.3(b). Furthermore, we get from Fig. 6(b) and Lemma 3.3(a) that the Q in Banach's definition is isomorphic to $\Delta^l \coprod \Delta^c \coprod \Delta^r$ since the coproduct operator is associative. Conversely, you can easily see that the Q of an injective inward production can be partitioned such that the resulting Δ-production is equivalent to the double-pushout production. This means that we can implement the double-pushout approach (at least in the case of injective productions) storing only one graph, namely Q, together with its partition.

A closer look to Fig. 6(a) allows generalizing Def. 3.2 such that noninjective redices are allowed. Since Lemma 3.3(a) does not assume q' itself to have a coproduct complement, but only $q' \cdot \bar{p}$, it is sufficient to require $g^l[\Delta^l \coprod \Delta^c] =$

$$
\begin{tikzcd}
\Delta^l \arrow[r] \arrow[dr] & \Delta^l \coprod \Delta^c \arrow[d, "g^l"'] & \Delta^c \arrow[l] \arrow[r] \arrow[d, "g"] & \Delta^c \coprod \Delta^r \arrow[d, "g^r"] \\
& G = \Delta^l \coprod \Delta^c \coprod R & \Delta^c \coprod R \arrow[l] \arrow[r] & \Delta^c \coprod \Delta^r \coprod R = H \\
(a) & & R \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \Delta^c \arrow[r] \arrow[d] & \Delta^c \coprod \Delta^r \arrow[d] & \Delta^r \arrow[l] \arrow[dl] \\
(b) & \Delta^l \coprod \Delta^c \arrow[r] & \Delta^l \coprod \Delta^c \coprod \Delta^r &
\end{tikzcd}
$$

Fig. 6: Δ-production and the double-pushout approach

$g^l[\Delta^l] \coprod g^l[\Delta^c]$. From this, we immediately get that Kaplan's formulaes (Def. 3.2) still hold if g^l is injective only on Δ^l.

A third advantage of discussing Δ-derivability in the categorical framework is that we can apply it to hierarchically labelled graphs, i.e., graphs the nodes and edges of which are labelled with graphs again. (In [5], we have shown how to construct pushouts in such a category.) In this case, we have to partition the labels in the center, too. Of course, we have some problems to draw larger graphs in the Δ-version; using colours can solve the problem. In a forthcoming paper, we discuss this technique in context of actor grammars; in this case it is of interest to label the nodes with term graphs.

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